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A derivation of the Prasad–Sommerfield solution

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Abstract. A certain form for the self-dual solutions of the Yang–Mills–Higgs system is tested. (When restricted to spherical symmetry this ansatz is the most general form in the Wigner–Eckart sense.) It is found that the only consistent solutions of this form are spherically symmetric, and that the only finite energy solution then is the Prasad–Sommerfield solution.

The Euler–Lagrange equations of the Yang–Mills–Higgs system with no self-interaction (potential) term of the Higgs fields are

$$\begin{aligned} &D_i F_{ij} + [\phi, D_i\phi] = 0 \\ &D^2\phi = 0, \qquad \varepsilon_{ijk} D_i F_{jk} = 0, \end{aligned} \tag{1}$$

where both the gauge field $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ and the Higgs field ϕ take their values in the algebra of the group SU(2). The last member of (1) is the Bianchi identity.

These equations are solved (Bogomolnyi 1976) by the following self-duality conditions

$$F_{ij} = \pm \varepsilon_{ijk} D_k \phi. \tag{2}$$

While (1) is a second-order system of differential equations, (2) is only first order and hence easier to handle. This is exactly the same situation as that occurring for Instanton (Jackiw *et al* 1976) solutions of the Yang–Mills field equations.

Here we seek solutions of (2), of the following form†

$$\boldsymbol{A} = \boldsymbol{\tau} \times \boldsymbol{\nabla} \ln \Theta + \boldsymbol{\tau}\Lambda \tag{3a}$$

$$\phi = \boldsymbol{\tau} \cdot \boldsymbol{\nabla} \ln \Theta\Omega, \qquad \boldsymbol{\tau} = -\tfrac{1}{2}\mathrm{i}\boldsymbol{\sigma}. \tag{3b}$$

Starting from this ansatz, we learn the following about the solutions of the self-duality equations (2):

(i) that solutions of the form (3) must be spherically symmetric. This conclusion is arrived at by starting with the *assumption* of axial symmetry, that is with Θ, Ω and Λ independent of the azimuthal angle, and then finding that the self-duality equations impose further the independence of Θ, Ω and Λ of the polar angle, and

(ii) that the spherically symmetric solution in question is the Prasad–Sommerfield (1975) (P–S) solution.

† A more special ansatz was considered by Manton (1978) where $\Omega = 1$ and Λ = constant.

Substitution of the ansatz (3) into (2) gives the anti-self-duality equation

$$\delta_{ij}(\Theta^{-1}\Delta\Theta+\nabla \ln \Theta\cdot\nabla \ln \Omega+\Lambda^2)+\partial_i\partial_j \ln \Omega-\partial_i \ln \Omega\partial_j \ln \Theta-\varepsilon_{ijk}\Lambda\partial_k\Omega\Lambda=0. \qquad (4')$$

Our procedure is to let the functions Θ, Ω, Λ depend on the two axial variables $\rho = r \sin \theta$ and $z = r \cos \theta$, and be independent of the azimuthal φ variable. We then show that the only possible solutions of (4) are those where Θ, Ω, Λ depend on the variable $r=(\rho^2+z^2)^{1/2}$ *only*, that is only spherically symmetric solutions.

Contracting (4) with ε_{ijk}, we find that $\Lambda = \text{const. } \Omega^{-1}$, and calling this constant k, (4′) becomes

$$\delta_{ij}(\Theta^{-1}\Delta\Theta+\nabla \ln \Theta\cdot\nabla \ln \Omega+k^2\Omega^{-2})+\partial_i\partial_j \ln \Omega-\partial_i \ln \Omega\, \partial_j \ln \Theta=0, \qquad (4)$$

which in terms of the following two functions

$$\chi_\rho(\rho, z)=(\partial/\partial\rho) \ln \Omega, \qquad \chi_z(\rho, z)=(\partial/\partial z) \ln \Omega$$

results in the following equations

$$(\Theta^{-1}\Delta\Theta+k^2\Omega^{-2})+\chi_z(\partial/\partial z) \ln \Theta+\tfrac{1}{2}\chi_\rho(\partial/\partial\rho) \ln \rho\Theta+\tfrac{1}{2}(\partial/\partial\rho)\chi_\rho=0, \qquad (4.1)$$

$$(\Theta^{-1}\Delta\Theta+k^2\Omega^3)+(\partial/\partial z)\chi_z+\chi_\rho(\partial/\partial\rho) \ln \Theta=0, \qquad (4.2)$$

$$(\partial/\partial\rho) \ln (\chi_\rho/\rho\Theta)=0, \qquad (\partial/\partial\rho) \ln(\chi_z/\Theta)=0, \qquad (\partial/\partial z) \ln(\chi_\rho/\Theta)=0, \qquad (4.3\text{–}4.5)$$

$$(\chi_\rho(\partial/\partial z)-\chi_z(\partial/\partial\rho)) \ln \Theta=0. \qquad (4.6)$$

From (4.3–4.5) respectively, it follows that

$$\chi_\rho=c_1f_1(z)\rho\Theta, \qquad \chi_2=c_2f_2(z)\Theta, \qquad \chi_\rho=c_3f_3(\rho)\Theta \qquad (5.3\text{–}5.5)$$

where $f_1(z)$, $f_2(z)$ and $f_3(\rho)$ are arbitrary functions, and c_1, c_2 and c_3 are arbitrary constants of integration. Comparing (5.3) and (5.5) we find that

$$f_1(z)=(c_3/c_1)\rho^{-1}f_3(\rho)$$

meaning that $f_1(z)$ is equal to a constant, say c. Then

$$(\partial/\partial\rho) \ln \Omega=\chi\rho=c_1c\rho\Theta=a\rho\Theta. \qquad (6.1)$$

Substituting (5.4) and (6.1) into (4.1) and (4.2), and subtracting the last two we get a simple equation for $f_2(z)$, which is then integrated†

$$f_2(z)=(a/c_2)z=bz$$

so that

$$(\partial/\partial z) \ln \Omega=\chi_z=bz\Theta. \qquad (6.2)$$

Using (6.1) and (6.2), the remaining equations (4.1) and (4.6) give

$$\Delta\Theta+k^2\Omega^{-2}\Theta+\tfrac{1}{2}bz(\partial\Theta^2/\partial z)+\tfrac{1}{2}a\rho(\partial\Theta^2/\partial\rho)+a\Theta^2=0 \qquad (7)$$

$$bz(\partial\Theta/\partial\rho)-a\rho(\partial\Theta/\partial z)=0, \qquad (8)$$

† We have put the integration constant c_4 in $f_2(z)=bz+c_4$ equal to zero. The only effect of leaving this constant non-zero is to give the same solutions, with respect to an origin translated by a distance (c_4/b) along the z axis.

and by repeated use of (8), (7) can be written in the following two equivalent forms

$$(\partial/\partial z)[(z^2+a^2\rho^2/b^2)z^{-1}(\partial\Theta/\partial z)+(2a/b-1)\Theta$$
$$+\tfrac{1}{2}(z^2+a^2\rho^2/b^2)b\Theta^2-\tfrac{1}{2}k^2\Omega^{-2}]+(a-b)z\Theta^2=0 \qquad (7.1)$$

$$(\partial/\partial\rho)[(\rho^2+b^2z^2/a^2)\rho^{-1}(\partial\Theta/\partial\rho)+b\Theta/a+\tfrac{1}{2}(\rho^2+b^2z^2/a^2)a\Theta^2-\tfrac{1}{2}k^2\Omega^{-2}]=0. \qquad (7.2)$$

We next notice that the constraint (8) can be thought of as

$$\frac{\partial\Theta}{\partial s}=\frac{\partial\Theta}{\partial\rho}\frac{\partial\rho}{\partial s}+\frac{\partial\Theta}{\partial z}\frac{\partial z}{\partial s} \qquad (8')$$

where s is a variable defined in terms of ρ and z. This implies that Θ is independent of s, and it is easy to see that it then depends only on the variable

$$t=(a\rho^2+bz^2)^{1/2},$$

which defines ellipsoidal surfaces on which Θ does not change for given t, and hence such solutions, if they exist, will be *axially symmetric.*

This restriction now leads to the following form of (7.1) and (7.2)

$$b(\mathrm{d}/\mathrm{d}t)[t\dot\Theta+(2a/b-1)\Theta+\tfrac{1}{2}t^2\Theta^2-\tfrac{1}{2}k^2\Omega^{-2}]$$
$$+(a-b)a\rho^2(\mathrm{d}/\mathrm{d}t)[t^{-1}\dot\Theta+\tfrac{1}{2}\Theta^2]+(a-b)t\Theta^2=0 \qquad (7.1')$$

$$a(\mathrm{d}/\mathrm{d}t)[t\dot\Theta+b\Theta/a+\tfrac{1}{2}t^2\Theta^2-\tfrac{1}{2}k^2\Omega^{-2}]-(a-b)bz^2(\mathrm{d}/\mathrm{d}t)[t^{-1}\dot\Theta+\tfrac{1}{2}\Theta^2]=0. \qquad (7.2')$$

In (7.1′, 2′) the function Ω is also taken to depend only on t, which follows from (6.1, 2).

We now subtract (7.2′) from (7.1′) and find that either $k^2=0$ or $\mathrm{d}\Omega/\mathrm{d}t=0$, which means that *either* Λ in (3*a*) vanishes, *or* Ω in (3*b*) is a constant. The second possibility is the one investigated by Manton (1978), and gives rise to the P–S in a complex gauge. Here we consider the $k^2=0$ ($\Lambda=0$) case.

Differentiating (7.1′, 2′) with respect to the variable s (which moves along the $t=$ constant curves) an inconsistency between the two equations arises. This inconsistency can be eliminated either by letting

$$a=b$$

or by letting

$$(\mathrm{d}/\mathrm{d}t)(t^{-1}\dot\Theta+\tfrac{1}{2}\Theta^2)=0. \qquad (9)$$

The second possibility (9) leads to the solution

$$\dot\Theta=\tfrac{1}{2}t[f_4(s)-\Theta^2] \qquad (10)$$

where $f_4(s)$ is an arbitrary function of integration. Substituting (10) into (7.1′, 2′) leads in both cases to

$$\Theta^2(t)=[(2a+b)/b]f_4(s) \qquad (11)$$

which means that both Θ and $f_4(s)$ are constants. This is a trivial solution.

We are therefore forced to revert to the other possibility where $a=b$, which leads to

$$t=a(\rho^2+z^2)^{1/2}=ar.$$

This proves that non-trivial (finite energy) solutions of the form (3) to the equations (2) must be spherically symmetric.

Under the circumstances, equations (7.1′, 2′) reduce to

$$r\dot{\Theta}+\Theta+\tfrac{1}{2}r^2\Theta=\text{constant}, \tag{12}$$

which, in terms of $\chi=r\Theta$, is

$$\dot{\chi}+\tfrac{1}{2}\chi^2=\text{constant}. \tag{12}$$

This being a real equation, the constant is real, so

$$\dot{\chi}+\tfrac{1}{2}\chi^2=0 \tag{12a}$$

$$\dot{\chi}+\tfrac{1}{2}\chi^2+\tfrac{1}{2}\lambda^2=0 \tag{12b}$$

$$\dot{\chi}+\tfrac{1}{2}\chi^2-\tfrac{1}{2}\lambda^2=0. \tag{12c}$$

Integrating (12*a*,*b*,*c*) we get, respectively:

(i) $$\chi=2/(r+a) \tag{12a'}$$

where a is an integration constant. The finite energy condition in this case

$$D_i\phi=\tau_j\left[\frac{1}{r+a}\left(\frac{1}{r}-\frac{1}{r+a}\right)\delta_{ij}-\frac{1}{r}\left(\frac{1}{r}+\frac{1}{r+a}\right)\hat{r}_i\hat{r}_j\right]\xrightarrow[r\to\infty]{}0 \tag{13a}$$

is satisfied but the energy density

$$\mathrm{Tr}(D_i\phi)^2=-\frac{1}{4}\left(\frac{4}{r^2(r+a)^2}-\frac{4}{r(r+a)^3}+\frac{3}{(r+a)^4}+\frac{1}{r^4}\right) \tag{14a}$$

is too singular to give a finite energy. This solution is not acceptable.

(ii) $$\chi=-\lambda\tan\tfrac{1}{2}\lambda(r+a) \tag{12b'}$$

in which we shall set the constant a equal to zero, as it has no effect on the following conclusions. Again, we compute the covariant derivative of ϕ

$$D_i\phi=\tau_j\left\{(\lambda\cot\lambda r-1/r)\frac{\lambda}{\sin\lambda r}\delta_{ij}+\left[\frac{1}{r^2}-\frac{\lambda}{\sin\lambda r}\left(\lambda\cot\lambda r-\frac{1}{r}+\frac{\lambda}{\sin\lambda r}\right)\right]\hat{r}_i\hat{r}_j\right\} \tag{13b}$$

which oscillates violently at spatial infinity. The energy density in this case is

$$\mathrm{Tr}(D_i\phi)^2=-\frac{1}{4}\left(2\lambda^4\frac{\cot^2\lambda r}{\sin^2\lambda r}+\frac{\lambda^4}{\sin^4\lambda r}-4\lambda^3\frac{\cot\lambda r}{r\sin^2\lambda r}+\frac{1}{r^4}\right) \tag{14b}$$

which is in fact regular at the origin, however its integral with respect to the volume element $2\pi r^2\,\mathrm{d}r$ is divergent, reflecting the fact that the finite energy condition is not satisfied, namely that (13*b*) does not vanish at infinity. This solution too is unacceptable.

(iii) $$\chi=\lambda\tanh\tfrac{1}{2}\lambda(r+a) \tag{12c'}$$

which is the P–S, and

$$D_i\phi=\tau_j\left\{\frac{\lambda}{\sinh\lambda r}\left(\lambda\coth\lambda r-\frac{1}{r}\right)\delta_{ij}+\left[\frac{1}{r^2}-\frac{\lambda}{\sinh\lambda r}\left(\lambda\coth\lambda r-\frac{1}{r}+\frac{\lambda}{\sinh\lambda r}\right)\right]\hat{r}_i\hat{r}_j\right\} \tag{13c}$$

which vanishes both for $r\to\infty$ and $r\to 0$, that is, it satisfies the finite energy condition and its energy density is regular at the origin. Integrating the energy density

$$\operatorname{Tr}(D_i\phi)^2=\frac{1}{r^4}-4\lambda^3\frac{\coth\lambda r}{r\sinh^2\lambda r}+2\lambda^4\frac{\coth^2\lambda r}{\sinh^2\lambda r}+\frac{\lambda^4}{\sinh^4\lambda r} \qquad (14c)$$

one gets the result 4π.

Thus the only finite energy solution of equations ($12a,b,c$) is the P–S, except that these being Riccati equations, given a solution χ_1, the function $\chi_1(r)+\psi^{-1}(r)$ is also a solution provided that ψ is given by

$$\dot\psi-\chi_1\psi=\tfrac{1}{2}. \qquad (15)$$

Substituting ($12c'$) for χ_1 in (15) we find

$$\lambda\psi=\cosh\tfrac{1}{2}\lambda r(\exp(\tfrac{1}{2}\lambda r)+\mu\cosh\tfrac{1}{2}\lambda r) \qquad (16)$$

where μ is a constant of integration. This leads to the solution

$$\chi=\frac{(1+\nu)\exp(\frac{1}{2}\lambda r)-\nu\exp(-\frac{1}{2}\lambda r)}{(1+\nu)\exp(\frac{1}{2}\lambda r)+\nu\exp(-\frac{1}{2}\lambda r)}; \qquad \nu=\tfrac{1}{2}\mu. \qquad (17)$$

This solution satisfies the finite energy condition and in fact becomes identical with P–S at large r, but the energy density function corresponding to it is singular at the origin and gives infinite energy.

Similarly, solutions generated in this way from ($12a',b'$) also result in infinite energy and are therefore also unacceptable.

In conclusion, we see that the only self-dual solutions of the form (3) are spherically symmetric, and that amongst these the only one with finite energy is the P–S solution.

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